TREATMENT OF THE CARTOGRAPHIC LINE

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ABSTRACT Traditionally, cartographic generalization has been considered in the context of eliminating map detail. In recent situations, however, details have been introduced to enhance the map message, as an aid in visual recognition. Enhancements are increasingly prevalent in displays of computer-generated lines, surfaces, and landscapes: some of the better known examples of this technique are fractal representations. The justification cited for this enhancement centers on improving the efficiency with which such displays may be constructed; the arguments against it focus on the visual characteristics which a map reader expects to see. A reconsideration of trends in cartographic research is presented, to explain how the change in generalization from 'elimination' to 'enhancement' has developed.

To the extent that geographic research is concerned with differentiations between information and error, the role of the cartographer focuses on maintaining a respectful balance between the two in graphic display. In the jargon, the intention is to 'maximize the clarity of the intended map message' by careful attention to what part is essential map information and what part is visual or conceptual noise. In cartographic line generalization, the distinctions between information and noise are in some cases quite subtle.

Most commonly, line generalization has been seen as the elimination of map detail. In recent situations, however, the introduction of detail has been used to enhance the map message as an aid in visual recognition. The justification cited for this enhancement centers on improving the visual characteristics which a map reader expects to see. This technique is increasingly prevalent in displays of computer-generated lines, surfaces, and landscapes. A reconsideration of the various methods for generalizing and representing cartographic lines may serve to explain how the change of emphasis from 'reduction' to 'introduction' of detail has developed.

Certain trends have developed over time: most notable are the trend towards quantification, and measuring lines as opposed to merely describing them; the shift away from determinism, in considering not only the line itself but the graphic region immediately surrounding it; and the trend to view generalization as a process including perceptual as well as mathematical components. These three trends have developed to a great extent from the research of other disciplines, ranging from geomorphology to political science. It is important for the reader to keep in mind that a certain amount of background information is necessary to review these developments, if at times the discussion seems to range far afield of the topic at hand.

It is also important to realize that the cartographic line has been treated within a variety of geometric contexts, as a set of points, a line, and even as an areal feature. Other researchers work within a spectral model, in which a cartographic line is broken into components of frequency, amplitude, and phase. The reason for this variety seems closely tied to the cartographer's attempt to cope objectively with a basically inductive task, namely, retaining the character of a geographic feature as it is represented at various cartographic reductions. What should become readily apparent in the course of this review is the difficulty involved in

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preserving something (namely the 'visual character' of the line) which has not yet been comprehensively defined.

THE LINE AS A SET OF EQUIPROBABLE POINTS

One of the earliest attempts at objectivity in generalization proposed a formula by which to compute the amount of information to discard for a given scale reduction. In the Radical Law, Töpfer and Pillewizer (1966) provided a mathematical rule formulating a geometric relationship between the amount of information contained in a feature (a river, for example, or a coastline) to the scale at which that feature is represented. For lines, information is defined by coordinate locations. The Radical Law states that when scale is reduced, the information (i.e., number of coordinates) should be reduced in proportion to the square root (hence 'radical') of the scale change. These constraints on information can be included to control the reduction according to the cartographer's choice of map purpose. Without going into the mathematics, one can say that the rule works, in practice. But an homogeneous information field is assumed along the length of the line, as each coordinate carries equal importance. The rule dictates how many points to remove, but not which ones.

Srnka (1970) derives a similar relationship, without the specification of scale. His reduction formula is thus suited to functional as well as scale generalization, although his discussions are limited to the latter type. The mathematics of this and the previous work indicate guidelines for the exponential decrease of information with linear decreases in scale. This kind of exponential relationship will turn up again and again, albeit in different forms.

Automated algorithms which treat the line as a set of equally important points include all of the techniques which fall within the realm of 'coordinate weeding,' or removing certain coordinates on the basis of some consistent criteria. Some examples of this include removal of every n^{th} point along a line, or removal of points selected at random; sampling points along a line at some even interval is another possibility (Lang 1969). In these algorithms, the probability that a coordinate will be removed or retained is dependent on its location in the sequence of points, rather than on its location in the context of a line feature. Thus, the coordinate which anchors the terminus of a sand spit has as much chance of removal as any point along the length of the spit. It is easy to understand how these algorithms could produce drastic (if unintentional) modifications of geographic shapes.

Smoothing a set of points can be accomplished as well, in treating a set of equiprobable points. Koeman and Van der Weiden (1970) suggest application of a simple moving average, in which some or all of the original points may be replaced. They compare the results achieved by exclusive runs and by overlapping runs of the moving average. In these techniques, as in coordinate weeding, no emphasis is placed on constraining the generalization according to specific line characteristics; a certain bias is implied by this, and also by the somewhat arbitrary choice of a starting point.

Boyle (1970) suggests an alternative solution, in preserving points on the line which are 'more important' to the accurate generalized representation. For examples, he cites submerged hazards on hydrographic charts, and the importance of accurately positioning their coordinates. These 'critical' points are not labelled as such, but are considered in a hierarchy by assigning weights (1–5) to each coordinate in the point set, to help a cartographer decide which point locations must be retained.

THE LINE AS A LINEAR FEATURE

In another geometric context, the line has been treated as a linear feature. It has no width, only length: one line can be distinguished from another by comparing curvature, 'wiggliness,' or sinuosity (Maling 1968). "The much more usual task of the geographer ... is to measure the length of an intricately sinuous line, such as the longitudinal profile of a stream, a stretch of coastline or even the road distance between two places" (Maling 1968, p. 148). Maling suggests that sinuous character can be measured by running a smoothed mathematical function through the line. But consider that most empiric or free-form curves in space cannot be easily represented with a single-valued function – the mathematics becomes too complicated to be of practical use. Parametric descriptions of these curves simplify matters, allowing the path of each component (in this case, x and y) to be expressed in turn.

For example, the graph A_s in Figure 1 appears simple, but it is difficult to describe one variable in terms of the other (i.e., $y = f(x)$ or $x = g(y)$). However, the curve can be expressed in parametric form ($f(s)$) by cubic equations, if each variable's path is described in isolation. Empiric lines such as rivers and coastlines are much more detailed that this simplistic example, of course. The value of parametric equations in describing approximations of geographic lines lies in their simplification of otherwise complex mathematical expression. It is for this reason that parametric equations are so often used in generalization by smoothed mathematical functions.

One method which automates parametric smoothing applies a spline function to the line. Classic splines are achieved by piecewise polynomial approximations of segments of the line, based on the derivative of each segment taken in turn. A large number of computations is involved, as a new polynomial equation must be derived for each segment in the spline. Chaikin's algorithm (1974) provides a somewhat faster computational alternative: he generates a curve by weighting the midpoints of a segment to generate a subsegment, then weights the midpoints of the subsegments, saving midpoints every time (Figure 2). Chaikin's algorithm is recursive, producing a sequential list of points to constitute the smoothed curve. It permits discontinuities as well as non-closure and self-intersection. Riesenfeld (1975) comments that Chaikin curves approximate a quadratic B-spline but are much more quickly constructed.

Another method which has gained recent popularity is the Bezier curve, developed originally for the automated milling machinery at Renault in France

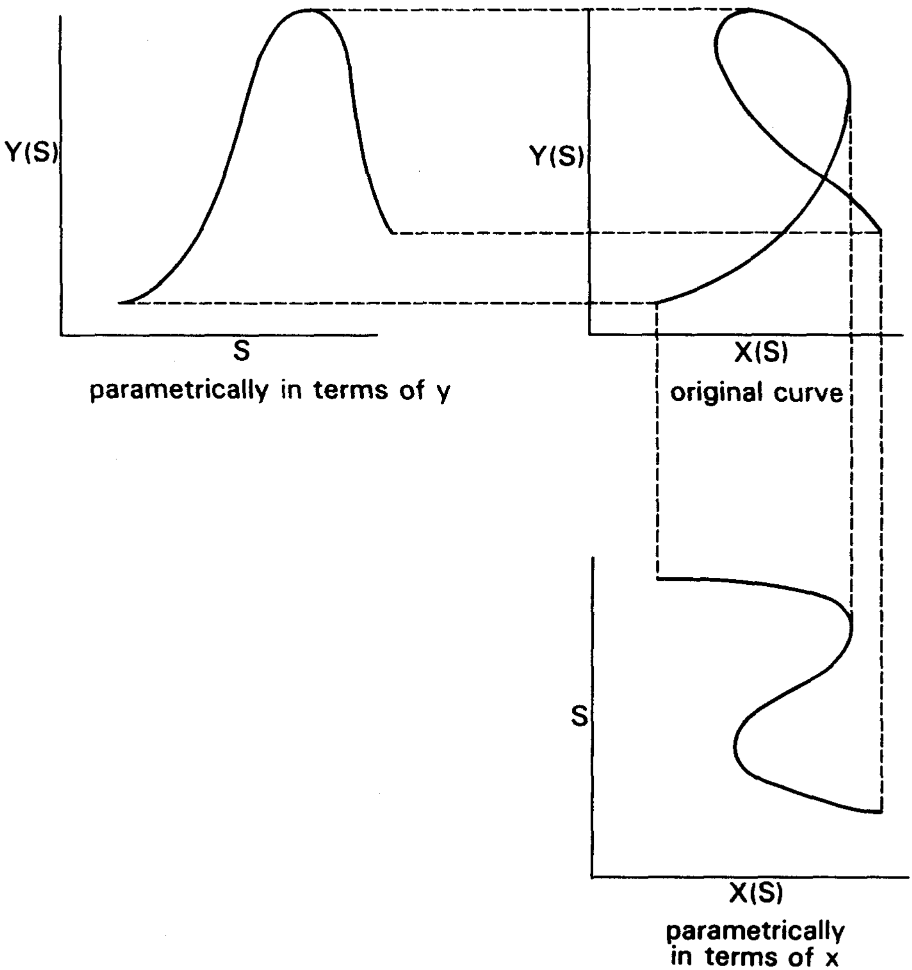

FIGURE *1*. *Re-expression of a complex curve.*

(Gordon and Riesenfeld 1974). This is another polynomial approximation, a series expansion of a binomial weighting function:

$$P(t) = \sum_{i=0}^{n} P_i \frac{n!}{i!\,(n-i)!} t^i(1-t)^{n-i}$$
$$= (1-t)^n P_0 + nt(1-t)^{n-1}P_1 + nt^2(1-t)^{n-2}P_2 + \cdots + t^n P_n$$

(taken from Chasen, 1978)

The series is expanded once for *x* and once for *y* coordinates, and the number of terms determines the number of points which will be generated. The resulting curve is similar in appearance to Chaikin's, but differs in computation. Chaikin computes midpoints for successively small segments, working from the center towards each segment endpoint. Bezier computes a probability surface which

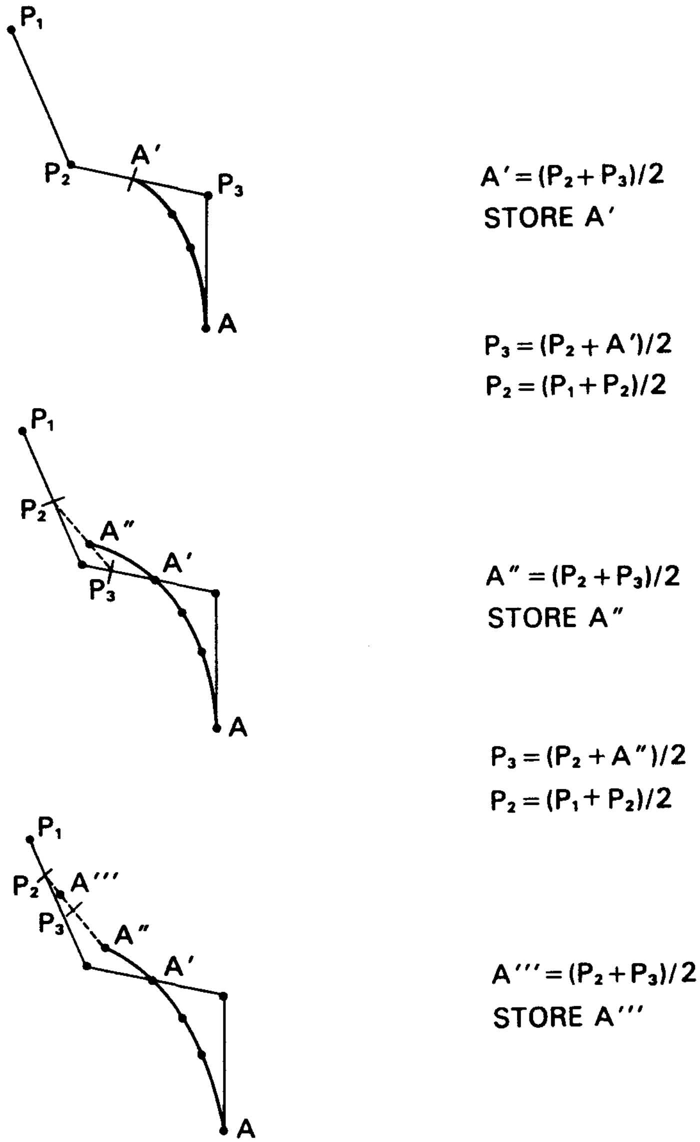

FIGURE 2. *Chaikin's algorithm (Chaikin, 1974).*

changes weights from one end of the segment to the other, effectively 'pulling' the smoothed curve towards the original as it moves along. Either of these algorithms would be adequate in Maling's application of a smoothed function to a cartographic line. The smoothed function may be used as a cartographic representation, or to compare the sinuosity of two different lines by measuring discrepancies in length or area between the two (Figure 3).

Two fundamental problems arise in this kind of treatment, as Maling (1968) himself admits. First is the question of which smoothing function to use. A moving

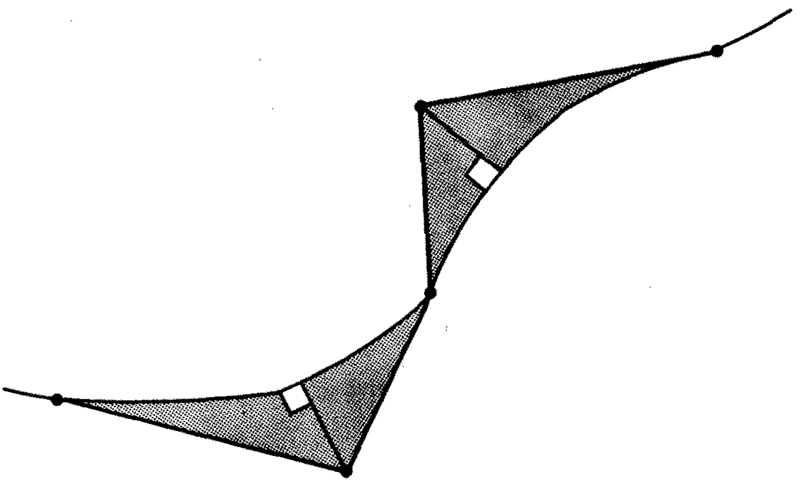

FIGURE 3. *Measuring discrepancy by distance and area.*

average, a spline function, and a Bezier curve will produce quite different visual representations of the same line, as an example. Furthermore, there is a problem of accuracy, as the smoothed function will contain only a small subset of the original coordinates. Second, and this follows directly from the first point, is the question of measuring the length of a geographic line. In order to determine distance or areal discrepancies between geographic features and their counterparts, one must first be able to define their lengths (or the lengths of their boundaries). For cartographic and geographic lines alike, this presents an interesting dilemma, which warrants a brief review.

DIGRESSION: PECULIARITIES OF GEOGRAPHIC LENGTH

Consider that the linear feature is continuous, and that the units used to measure its length are discrete. It turns out that the more precisely one measures an empirical (i.e., naturally occurring) line, the longer that line becomes. Several Europeans (Shokalsky 1930; Steinhaus 1954, 1960) pointed this out, that the series of lengths obtained by repeated measures using smaller and smaller units does not converge.

If the length of the Puget Sound coastline is measured on a LANDSAT image, its length will tally at some (rough) multiple of 79 meters, assuming that image pixels are 79 meters on a side. (This is not absolutely true; but it will serve for the purpose of the argument). Features of the coastline which are smaller than this will not be present in the image, and so will not be measured. A high-altitude (70,000 feet) photograph of this area will include some of these features, however; the length of the coastline in this image will equal the length as measured on the LANDSAT representation *plus* the length of all additional features resolved on the larger scale photograph. The coastline will therefore be longer, by definition. A low altitude (20,000 feet) photograph will include still more features; adding the length of these to the coastline measure will increase the length once again. And so on.

Table I VALUES OF α FOR SELECTED COASTLINES (AFTER RICHARDSON, 1961)

Britain	0.25	Australia	0.13
Germany	0.15	South Africa	0.02
Spain-Portugal	0.14		

The Polish mathematician Hugo Steinhaus suggested that a practical solution to the paradox of length would employ measurement techniques which define only a lower limit for the length of a geographic feature (Steinhaus 1954). He defined the length of an arc [L] as the limit of the summed lengths of straight line segments [$\Sigma l(i)$] making up the arc, as the segment size [l] becomes infinitely small:

$$L = \lim_{l \to 0} \sum_{i=1}^{n} l(i)$$

He maintains that this limit always exists; if infinite, however, the arc is said to be non-rectifiable. Steinhaus (1954) developed a 'longimeter' to approximate measures for the length of a curve lying in a plane, as follows. He cut the plane with a series of equidistant parallel lines, then counted the number of lines which intersect the curve. The measure achieved by this method is said to be 'length of order n,' where n equals the line spacing of the longimeter.

Other disciplines have expressed interest in this problem as well. Some of the best known work in the area was done by Lewis Richardson, who investigated the paradox of geographic length when sidetracked from political science research relating the length of political frontiers to the occurrence of national boundary disputes. He used map dividers at various separations to measure coastlines and rivers in political boundaries on atlas maps. He found that apparent boundary length increased steadily with the decrease in divider separation, and the rate of increase differed somewhat for each measured line (Richardson 1961). When repeating the measurement process for a regular polygon such as a circle, Richardson found that here, total length measurement stabilized quickly with decreasing units of measure.

Figure 4 shows a comparison of lengths for various coastlines, and for a circle, values which Richardson measured and plotted as logarithms. The negative trend of the plot led to the following derived relationship:

$$\Sigma l = l^{-\alpha}$$

Σl represents total measured length, and l represents the magnitude of the measurement unit, or divider separation. Richardson states "α is a positive constant, characteristic of the frontier" (Richardson 1961, p. 170). He goes on to say that α may be positively correlated with the perceived irregularities of the coastline, or its lack of smoothness, concluding that the differing slopes of his plots correspond to the characteristic irregularities of the coastlines they represent (Table 1).

Two facets of this work are important in looking at the development of cartographic line generalization: first are Richardson's notions about accuracy. He comments, "It is doubtful whether the total polygonal length of a seacoast tends to any limit as the side of the polygon tends to zero." (Richardson 1961, p.

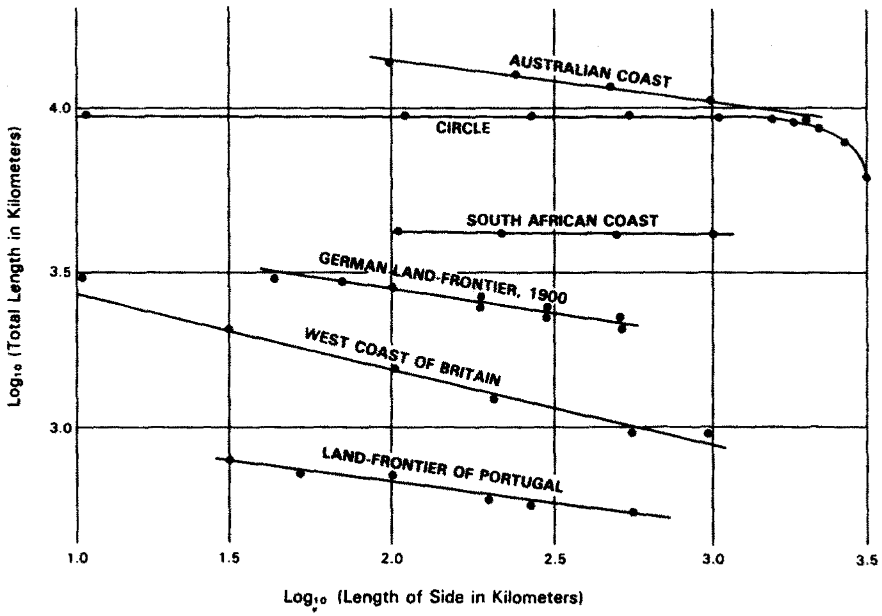

FIGURE 4. *Richardson's empirical data on the rate of increase of coastline's lengths (after Richardson, 1961).*

170). This is not to imply that the most accurate length is derived from the most precise measurement, but rather that accurate measurement becomes undefinable at the limits of precision (when unit length approaches zero). The problem of deciding which level of precision will produce the greatest degree of accuracy is left entirely to Richardson's readers – he is only pointing up the fallacy of over-reliance on precision in cartometric tasks. Chronologically, this work provides an early step in the shift away from determinism.

The second facet of the work which is important is the derivation of the exponent α, which is a preliminary quantification for the characteristic irregularities of a sinuous line. European research had also focused on the validation of map-measured distances (e.g., Shokalsky 1930). In these studies, too, regression-like equations were formulated by which 'actual' geographic length could be predicted, based on the particular units of measurement. All of the formulas include a term variously identified as a measure of line character, irregularity, or sinuosity; but this term was most often left as a by-product of the derivation. Richardson considered it a trivial finding; his interest remained focused on the accuracy of geographic measures. However, the topic (and the exponent) will be returned to further along in this discussion.

THE LINE AS AN AREAL FEATURE OF FINITE WIDTH

As discussed before, the Polish mathematician Steinhaus suggested that a practical solution to the paradox of length would employ measurement techniques which define only a lower limit to the length of a geographic feature. The

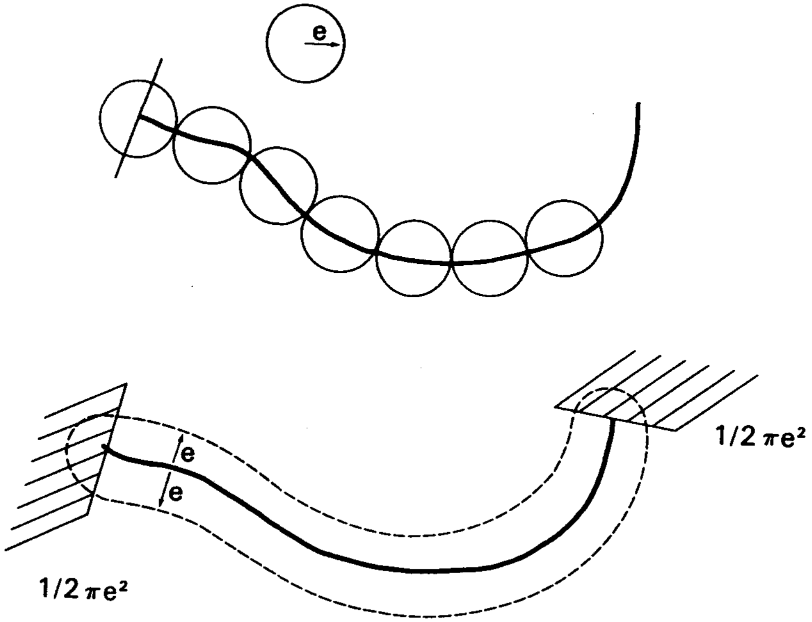

FIGURE 5. *Perkal's e-neighborhood.*

technique as developed by Perkal (1966a, 1966b) is directly applicable to the study of geographic process, and provides a good example of how a cartographic line may be treated as an areal region, a feature having width as well as length.

Perkal sidesteps the problems involved in measuring a (possibly infinite) geographic length, working instead to determine at what scale of measurement the length becomes finite, or rectible. He proceeds by constructing an ('e-neighborhood' 'e' for empiric) around a curve. The e-neighborhood is the set of all points on the place for which the distance from the curve is less than or equal to some small but finite value epsilon. The effect is similar to that of rolling a circle along a curve, letting the path of the circle define an envelope or smooth closed polygon surrounding the curve. Perkal then computes the area [A(e)] of the e-neighborhood, subtracting the area of one half circle [$\frac{1}{2}\pi e^2$] at each end (Figure 5). The length of the curve will be directly proportional to the area of the e-neighborhood, and inverse to the diameter 2e of the circular, thus:

$$L = \lim_{e \to 0} \frac{A(e) - \pi e^2}{2e}$$

The final step is to simplify the areal calculations, following from Steinhaus, replacing the area term with a simple count (n) of the number of (non-overlapping) circles of radius e which the curve intersects. He computes an 'epsilon length' for curve X as

$$L_e(X) = n - (\pi/2)e$$

The units of e-radius determine the units of epsilon length. Thus the accu-

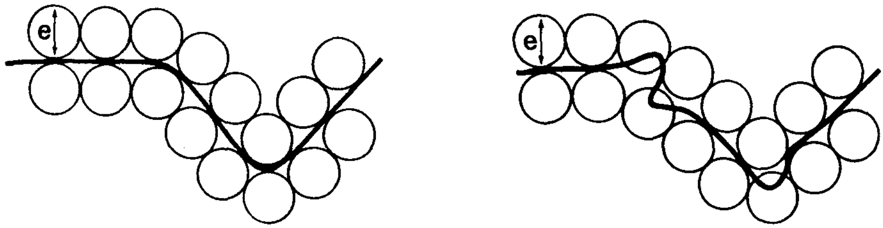

FIGURE 6. *e-convexity (after Perkal, 1966).*

racy of the measured length is in fact determined by the scale of the measurement. Notice that an insignificant change in epsilon will not change the epsilon length substantially: the measure is in this sense more robust than the length measures of either Steinhaus or Richardson. It is also interesting for the moment to consider **e** as a diameter instead of as a radius (Figure 6). A curve is said to be **e**-convex if the string of tangent circles (of diameter **e**) are tangent to the curve only at a single point. This curve on the left is called **e**-convex, for this particular value of epsilon; the curve on the right is not. However, the curve on the right *will* be **e**-convex for some lesser value of epsilon, which allows discernment of the smaller arcs in the curve. Perkal suggests using the envelope of the **e**-neighborhood as a generalized version of the empiric line. In a mechanical context, this is exactly what happens during some stages of the cartographic process.

Consider, for example, the minimum turning radius of a polar planimeter, a plotter head on a Calcomp flatbed, the magnetic grid size of a graphics tablet, or even the pixel size on a satellite image. These are effective **e**-values for the respective instruments: all lines plotted, all lengths measured will be a function of this value. It may be functional in a particular research design to specify a value of epsilon instead of relying on the tolerance thresholds of the machine. Maling (1968) considers the size of the white dot on the stereoplotter platen in this context. Boyle (1970) makes similar comments on the generalizing effects of hardware limitations in computer-drawn maps. Brophy (1973) designed a computer algorithm which effectively rolls a circle along a line, generalizing by removing details which fall within the boundaries inscribed by the circle.

Another interesting point here is that the **e**-neighborhood is not always symmetrical on both sides of the empiric line: that is in part due to the magnitude of epsilon. It is also a function of the sinuous twists and characteristic irregularities of empiric lines. And herein lies a direct application to the study of geographic process.

The boundary of the **e**-neighborhood may be thought of as the trace of points nearest the coastline left by the path of a circle of **e**-radius which is rolled along the coastline on both sides. Because of the shape of the capes and bays, the length of this trace will not be the same on each side of the seacoast. Ships swing wide to avoid the capes but trains swing inland to avoid bays and estuaries. The coastline, then, appears to have an inside length and an outside length for purposes of movement along its perimeter, depending upon the technical requirements of the vehicle used. This statement may be made more general. The boundaries of the **e**-neighborhood of a line are ordinarily not equal; the one-side length equals the other-side length only if the line is **e**-convex. *(Nystuen 1966, no page)*

Perkal's techniques provide a broad methodology by which to investigate the geographic notion of a linear boundary as region. Nystuen considers the effects which regional shape may have upon a spatial process: non-symmetry of the e-neighborhood may facilitate studies of a process which affects one side of the boundary but not the other, studies of flows across a boundary, and studies of processes which exist only at the boundary (i.e., within the e-neighborhood). Nystuen calls this application the concept of local convexity, or 'convexity-in-the-small.'

There are direct implications for line generalization as well. To return to the coastline example, a cartographer might generalize one side of the e-neighborhood of the California coast in mapping seasonal traffic flows along Highway 101, and/or generalize the other side of the coast to map fishing rights along the three-mile international boundary.

THE LINE AS A SPECTRAL FEATURE: FILTERS AND FREQUENCIES

Applications of Perkal's method to line generalization are for the most part not common in the literature; Nystuen's reference to e-neighborhoods has been for the most part neglected by cartographers. The generalization algorithms which were developed during the 1970s tended to focus on simplifying the existing line, rather than replacing coordinates by means of smoothing functions and splines. With the increasing prevalence of automation in map-making, cartographers could rely on improving the accuracy of their displays. Accordingly, the objective in generalization shifted somewhat: the purpose was not so much to reduce information, as it was to filter it. For the most part the filtering process was computerized by setting arbitrary tolerance limits within which points along the line could be eliminated. Two kinds of tolerancing were pursued. In each one, an envelope is constructed around the line, in a manner similar to Perkal's e-neighborhood.

In angular tolerancing, a cone of angle 20 is projected forward from each point along the line, in succession (Figure 7). Points ahead of the vertex are discarded, until a point is found which falls outside the threshold of the cone. The cone is then moved to this point, and the process repeated. The disadvantage of the algorithm is its tendency to give long, gradual curves a blocky appearance; because of this, the cone is usually constrained to a fairly short forward search.

Distance tolerancing is quite similar in nature, eliminating points on the basis of a threshold width for a corridor extended along either side of the line (Figure 8). This method also requires a limiting distance for forward searches, and is best suited to the elimination of small errors from sets of stream-digitized coordinates.

Tolerancing was formalized and refined by Peucker (1975) (now Poiker), who combined the notion of an areal band or corridor with a consideration of linear details which occur at a specific frequency. Each frequency has an associated bandwidth by which it may be represented; in the graphic adaptation of this electronic metaphor, decreasing bandwidths can be used to selectively isolate portions of fine detail (so-called high frequency information) from the line. The bandwidth model is operationalized as a data encoding scheme. Endpoints of the

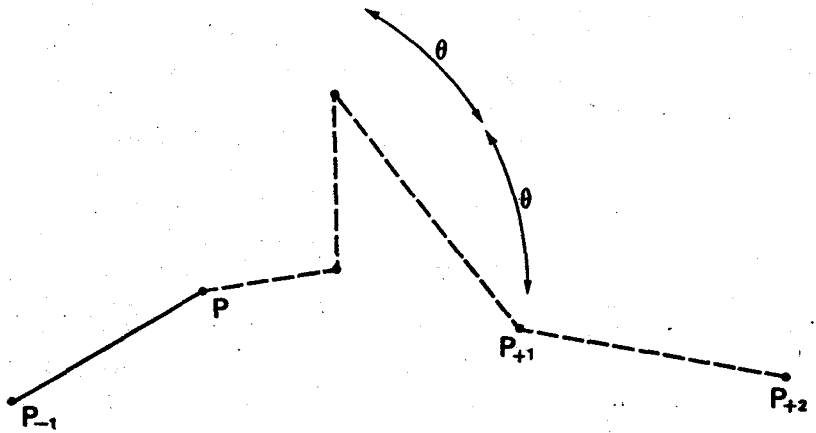

FIGURE 7. *Angular tolerancing algorithm.*

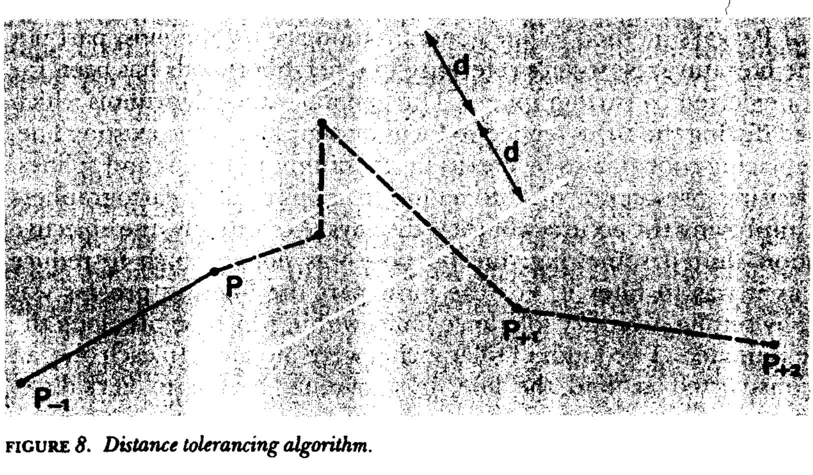

FIGURE 8. *Distance tolerancing algorithm.*

line form the length parameter for the band, and its width is defined by points of maximum deviation on either side of the straight line segment connecting the endpoints (Figure 9). These four points are used to define bandwidths of the next higher order frequency, as shown in the figure. The process of encoding continues until the desired minimum bandwidth has been achieved.

Applications of Poiker's work are common in geographic and computer science journals. Ballard (1981) has automated the bandwidth encoding scheme as a data structure which he calls Strip Trees. Poiker cites several cartometric applications for bandwidth encoding, including computation of line intersections, matching of digitized lines, and searching tasks within polygons. The importance of his work lies in the ability to tie the accuracy of the application to some finite bandwidth which encompasses a particular scale of representation.

The technique is also useful in line generalization, which in this context becomes a simple process of selective bandwidth filtering for high, medium, or

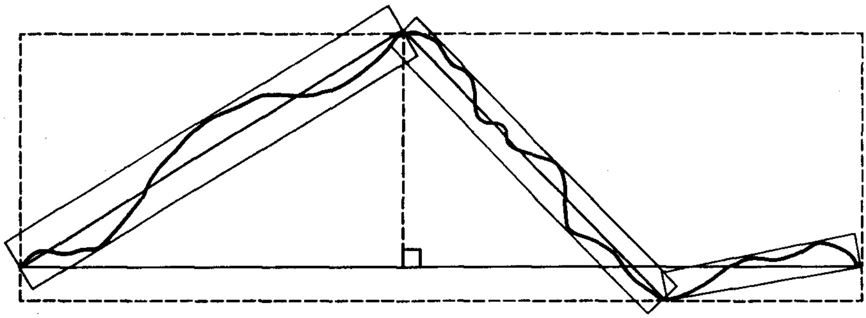

FIGURE 9. *Bandwidth encoding (after Peucker, 1975).*

low frequency details. In Douglas and Peucker (1973) was developed one of the few tolerancing algorithms to select coordinates, as opposed to eliminating them, based on ideas which were subsequently refined into this bandwidth model. (It is interesting to note that Ramer (1972) devised a similar technique at about the same time.) Although it seems a trivial distinction, there is a significant perceptual difference in a line generalized by selection of points: this distinction, and the algorithm which produces it, will be discussed in a later section of this paper.

An earlier discussion of generalization as a filtering process is provided by Tobler (1966). In considering cartographic information as occurring at various spatial frequencies, he views generalization as a filtering of components such as amplitude, wavelengths, and phase. Tobler suggests using Perkal's epsilon to identify threshold frequencies at which a geographic feature is sensitive to the filter. At these frequencies, smoothing takes effect. The filters he describes are matrices of probabilities, which are used as smoothing weights. For example, a binomial weighting function provides a high-frequency filter which has a strong generalizing effect on small details. However, a Fourier series provides another valid filter, and may be designed to smooth information at a variety of different frequencies.

As a matrix manipulation, the frequency response of a spatial filter is computable. Tobler has derived a concept of generalization as one more kind of transformation which is objective, predictable, and in some cases, reversible: as a matrix transformation can be computed, so can its inverse, if it exists.

> As a priori criteria one would expect that generalization ... should somehow eliminate 'small scale' features. If generalization is considered as a transformation, it is natural to inquire whether this process can be reversed. If topographical maps are generalized the result should conform to appropriately modified accuracy standards. Finally, certain statistical parameters ... should be preserved by the transformation. *(Tobler, 1966, p. 1)*

The importance of this work in the development of generalization becomes clear if all the threads of this research discussion are tied together. According to Steinhaus, and to Richardson, an empiric line can only be approximated in graphic representation because its geography can never be precisely measured. Thus any graphic representation provides at best only a generalized version of a linear feature. Perkal's approximations of lines by e-neighborhoods indicates that the set of information which is included in the generalized representation will be a function of the scale (measurement units), and the purpose of the measurement (which side of the line is measured). It will also be dependent on the places of e-convexity, on the sinuous curves characteristic of the line. Notice that at this time, sinuosity has still not been defined, although Richardson's work points to a possible index.

What Tobler has added to this thread of inquiry is that these generalized representations can be arrived at through a filtering process which in theory can be designed to remove or to restore information, to smooth or to enhance the graphic representation, in effect. He also establishes the importance of preserving statistical characteristics of the line as it is filtered in order to preserve the character of the representation. It seems appropriate at this point to digress once again for a brief review of one methodology which uses statistical parameters as a relative measure of line sinuosity.

DIGRESSION: LINES OF FRACTIONAL DIMENSION

Let us return for a moment to the work of Lewis Richardson (1961), who found that small decreases in unit size result in often substantial increases in measured length. The rate of increase was found to vary for specific coastlines which were measured. Richardson commented that these rates were possibly related to the sinuous nature of each coastline, but saw no great importance in his findings. It remained for Mandelbrot (1967) to derive its theoretical basis. His work also provides a comprehensive example of generating figures which retain certain statistical properties.

Mandelbrot began by reexpressing Richardson's relationship ($\Sigma l = l^{\alpha}$) as an equality between the increase (r) in total line length and the size ($1/N$) of N equal steps required to travel the length of the line.

$$r = (1/N)^{1/D} \qquad D >= 1.0$$

What this formula describes is really quite basic – if a straight line is broken into N equal pieces, the ratio of the length of any piece to the length of the whole line will be $1/N$. This is called the ratio of self-similarity: because it is linear, the figure is said to be empirically one-dimensional. In two dimensions, a plan can be subdivided into N equal squares (as in tiling). Each square will have an area of ratio $1/N$. In three dimensions, the similarity ratio will be in cube roots, or $(1/N)^{1/3}$. In general form, the value which D takes on is mathematically equivalent to the dimension of the feature.

Solving for D:

$$\begin{aligned} \ln r &= \ln [(1/N)^{1/D}] \\ \ln r &= 1/D \ln (1/N) \\ D \ln r &= - \ln N \\ D &= - \ln N / \ln r \end{aligned}$$

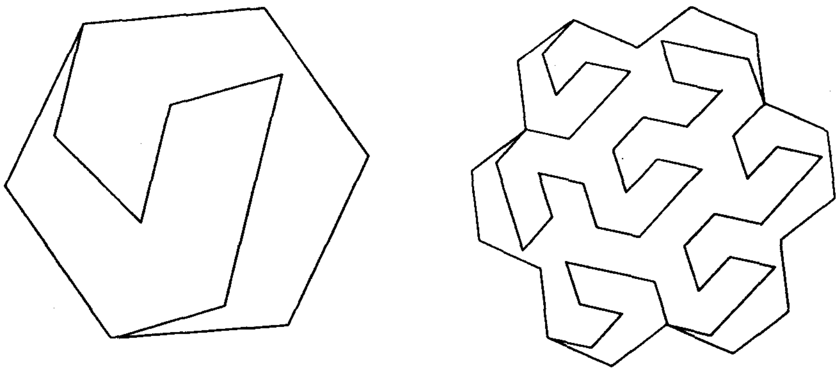

FIGURE 10. *Peano curve (after Mandelbrot, 1982).*

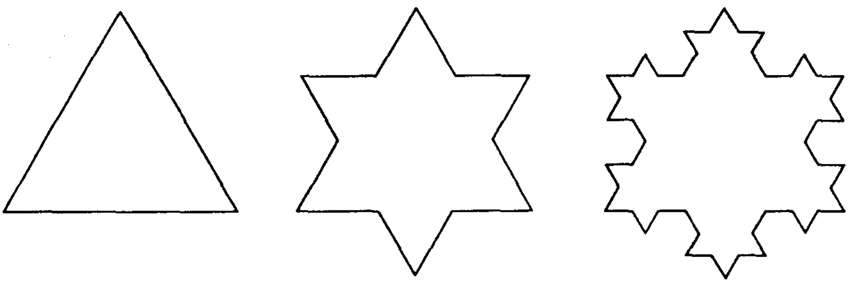

FIGURE 11. *Triadic Koch Island (after Mandelbrot, 1982).*

NOTE: substitutions between negative logarithms and logarithms of fractions will simplify arithmetic below; the computational formula is as follows:

$$D = \ln N / \ln (1/r)$$

As a computed ratio, D can take on fractional values: Mandelbrot's derivation expands the concept of 'dimension' into the realm of real numbers. Thus a figure will have a topologic dimension (as a point, line, area, and so forth) which describes to what class of figure it belongs; and it will have an empirically computed dimension which describes its 'degree of complication' (Mandelbrot 1967, p. 636), or the characteristic way in which the figure fills up the space which it occupies. D is conceptually equivalent to Richardson's (1961) exponent α in describing this property, in that it varies according to the irregularities of the line being measured. In contrast to Richardson, however, Mandelbrot does not concern himself with problems of absolute length, focusing instead on the rate of its increase. An example may serve to explain the concept of empiric dimensionality; and then attention can be returned to a more geographic application.

Take for example the mathematical curiosities such as Peano Curves (Figure 10) or Koch Triadic Islands (Figure 11), as they are constructed. As linear features, the curves are topologically one-dimensional. For the Koch Island, each side of the original unit triangle is replaced by a four-sided open polygon (the number four is arbitray: five, six, or N sides are also possible); thus $N = 4$. Notice that the length of this side has increased by 1/3 in making this replacement; thus $r = l3$.

$$D = \ln(4) / \ln(1/(1/3))$$
$$= \ln(4) / \ln(3)$$
$$= 1.2618$$

In effect, this value represents the rate at which the curve will fill the plane in which it lies. For the Peano curve, $D = 1.1291$, which means this curve will fill the plane less quickly than the Koch Island, for equal values of.... It should be obvious in looking at the two curves that this is so: D provides an index of the mathematical complexity of the figures, which is related to their visual sinuosity as well.

Mandelbrot calls these figures *fractals*, because their empiric dimension is fractional. Fractals differ from regular curves and polygons in that a regular geometric figure will have an empiric dimension which is an integer. As shown by the plot of the circle in Figure 4, the integer dimension reflects the fact that overall length of regular polygons stabilizes very quickly to some finite value, while the same cannot be said for fractals. Coastlines appear to have fractal characteristics, then, because their length does not stabilize: Mandelbrot (1967) computes the fractal dimensions for Richardson's coastlines to be 1.25 (Britain), 1.15 (Germany), and 1.02 (South Africa), concluding that Richardson's exponent is computable from the fractal dimension as

$$\alpha = 1 - D$$

For the Koch Triadic Island, then, each unit side of the figure is replaced by a four-segment curve. Each of these four segments becomes a new unit side in the resulting figure, and the process is repeated. As each segment in the curve is replaced by small scale replicas of the entire curve, the figure is said to become self-similar; and the fractal dimension provides a ratio describing the rate at which the self-similar replication process increases the length of the curve.

Mandelbrot maintains that the self-similarity concept has logical validity for coastlines as well as for mathematical constructions such as space-filling curves. The self-similarity for Peano curves and Koch Islands is termed 'precisely equivalent,' because any piece of the curve can produce an exact replica of the whole. (As an aside, a holographic image is precisely equivalent in an optical sense, in that if the holographic glass plate negative is shattered, any piece can be used to reconstruct the entire image (Fleischer 1982)).

On the other hand, statistical equivalence is probably a better term to describe the self-similarity of coastlines and other naturally-occurring features. Mandelbrot demonstrates this by generating a curve of random Brownian motion, and then generating a Brownian curve constrained by two different values of D (Figure 12). The stochastic process produces lines which bear some visual resemblance to coastal sinuosity; and further constraints produce other kinds of apparently geographic lines (Figure 13).

Dutton (1981) has developed an algorithm which produces fractal models of actual cartographic lines. The process involves standardizing angles according to four predetermined tolerance parameters. His intention is to introduce self-

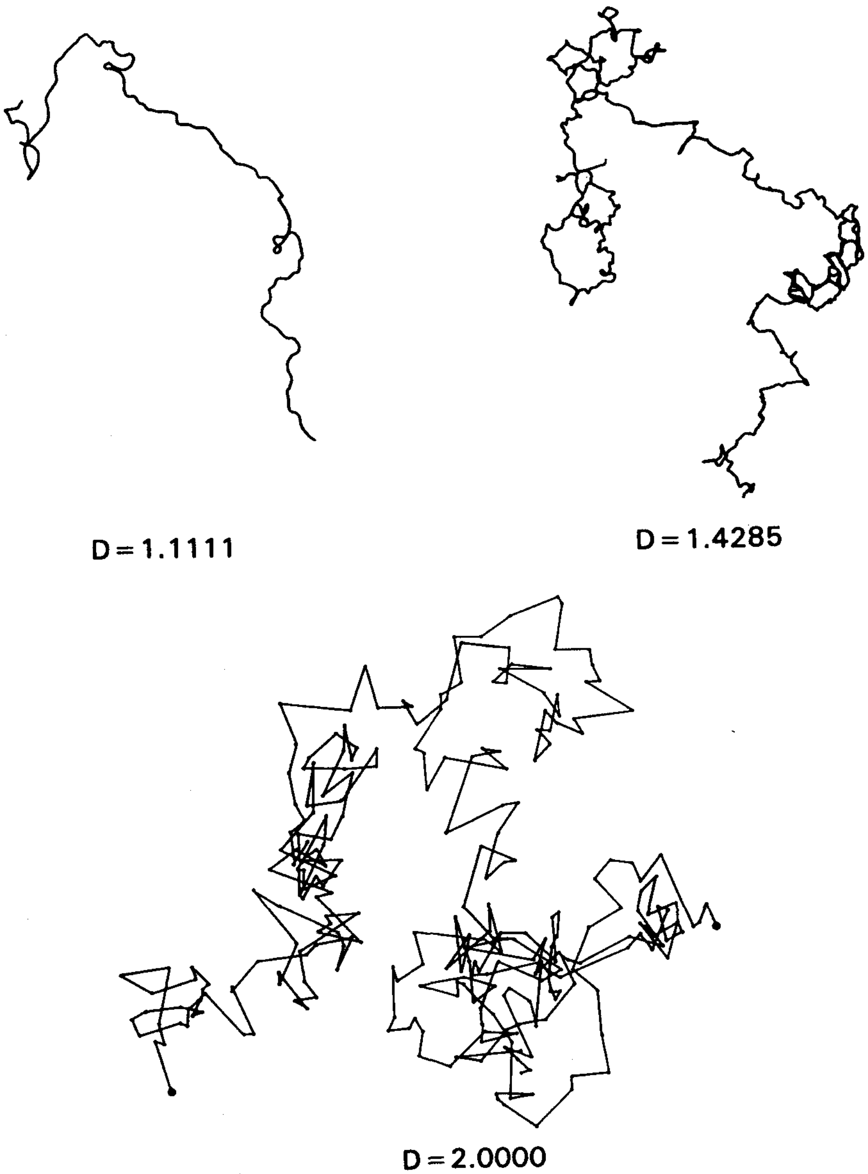

FIGURE 12. *Graphical meaning of D (after Mandelbrot, 1982).*

similarity to the representation, by standardizing its shape irregularities. "One wishes for a measure of geometric complexity and irregularity that is as general as that of entropy in thermodynamics. Fortunately, foundations for such a vocabulary and for such measures have been developed." (Dutton 1981, p. 24). Once the angles have been standardized, any portion of the line will exhibit the same complexity which characterizes the line as a whole: this is a pseudo-approximation of the self-similarity which Mandelbrot proposed.

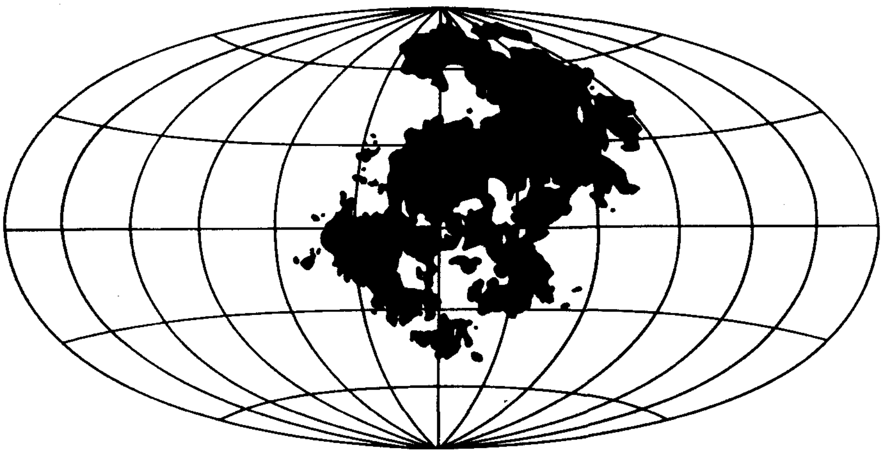

FIGURE *13*. *Stochastic pangaea (after Mandelbrot, 1982).*

The algorithm computes midpoints for two adjacent segments, connecting these to form a triangle with the common endpoint (Figure 14). At the apex of each triangle is the common endpoint – this is the vertex to be moved. The vertex is slid along the apex bisector until its angle becomes (nearly) equivalent to some predetermined value. Then midpoints of the two segments become fixed vertices, and the process is repeated. The geographic accuracy of the line is obliterated by moving the original vertices, of course, but this problem could be easily remedied by holding constant the vertex positions and moving only the midpoints.

Four parameters constrain the fractalization. The first parameter, SD, determines the angle of standardization which Dutton says is equivalent to Mandelbrot's fractal dimension (D). The second, UC, is a uniformity coefficient: ranging between −1.0 to +1.0, it determines the degree to which the angles are equivalent. At +1.0, verticies are moved the entire distance needed to produce equivalent angles; at +0.5, they are moved only half of the necessary distance, and so forth. For negative values, movement occurs in the opposite direction. Two other parameters respectively define the maximum and minimum applicable segment lengths to fractalize, controlling for straightness (ST) and smoothness (SM) in the final presentation. These are similar to the (distance and angular) thresholds used in the tolerancing algorithms discussed previously. For example, they would insure retention of long gradual curves during fractalization. Even with these last two, some smoothing is necessary subsequent to the enhancement process, to 'fine tune' the look of the line.

> Unlike [splining] and other methods for coordinate reduction and chain smoothing, fractalizing permits features to be exaggerated and smaller scale features to be introduced into digitized curves, as well as allowing features to be eliminated. The exaggerations and additions are not arbitrary forms introduced to the chain, but are caricatures and recursions of forms already found there. *(Dutton (1981), p. 25)*

Dutton's algorithm works to perserve linear character, then, by reexpressing shapes which are already present, and by introduction of standardized caricatures

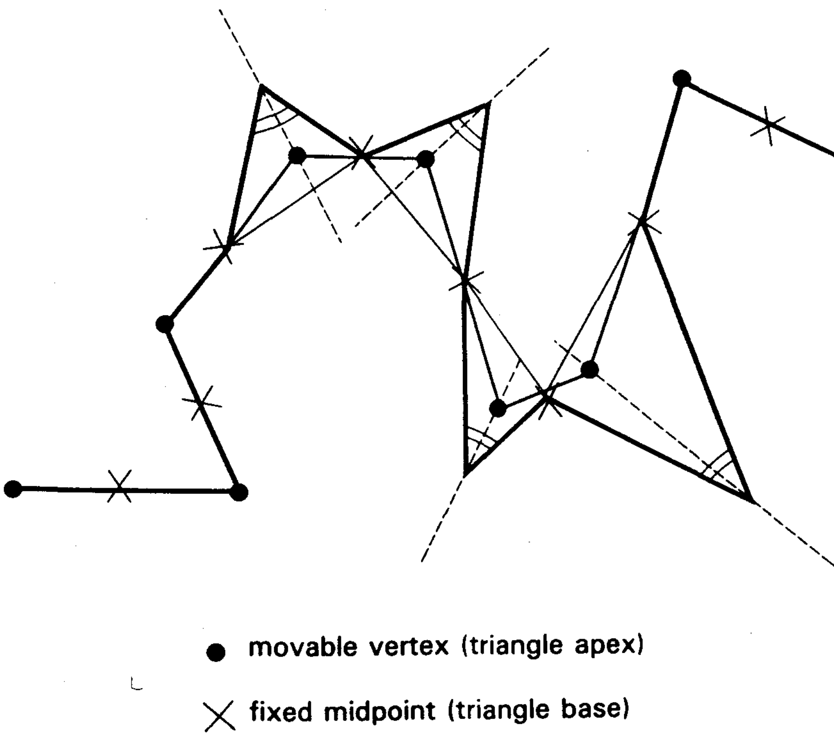

FIGURE *14*. *Dutton's fractalizing algorithm.*

at smaller and smaller levels of detail. He offers no guidelines for choosing the values of his four parameters to construct lines of varying character, except to say that his parameter SD is operationally equivalent to the fractal dimension. But as Dutton admits, the algorithm is quite sensitive to localized deviations in sinuosity, which are not accounted for by the constant value of SD. And so while it has been established that linear character can be mimicked, a formalized definition has still evaded both manual and algorithmic attempts.

The foregoing discussion does not imply that fractals can provide geographically accurate representations, but rather that fractal constraints on stochastic motion may be useful in modelling some of the visual components which are present in geographic representations. In other words, what you see in a fractal image is similar to what you expect to see in a geographic representation – the visual character of the two images are quite similar. The computer graphics industry has capitalized on this effect, using fractal geometry to generate all sorts of generic landscape objects, including clouds, trees, and even mountain range backdrops. "The major problem in making a realistic computer animation is capturing the right balance between order ... and disorder." (Fleischer 1982, p. 52). It is the randomness of Brownian motion which has been most commonly used to generate fractal landscapes which appear to be realistic geographic representations (Carpenter 1981; Fournier & Fussell 1980).

This approach has been criticized in the literature, most notably for the fallacy of considering geographic features to be self-similar. Goodchild (1980) comments that geographic features are generated by geomorphic processes which are scale-dependent in nature, and which supply visual clues to help determine the scale of the feature in representation. These clues are not present in a line or a surface which is its own replicate at every scale. Goodchild adds that this does not apply to, for example, lunar landscapes, whose form is related to very different morphogenic processes. This may be the reason why fractalized landscapes appear so 'other-wordly' (Whitted 1982).

Derivation of the fractal dimension has not resolved the paradox of empiric length, but has provided a numeric index for lines of relative 'complicatedness' (Mandelbrot 1967), 'irregularity' (Richardson 1961), and/or 'sinuosity' (Maling 1963, 1968). Time and again, researchers have come up against the problem of distinguishing between lines, and between linear characteristics, in an objective manner. Until visual character can be analytically identified, it will remain a major stumbling block in automating generalization tasks.

It should be obvious, however, that attachment of a simple metric to a set of geographic lines is not sufficient as an analytic distinction of line character; and this is especially true in a cartographic context, where the visual effect of a display can change quite drastically with changes in scale, projection, resolution, and so forth. A review of pertinent research on the perception of linear information should provide a perspective on the map reader's view of a generalized line, and should lead to a clearer definition of line character in generalization.

THE LINE AS A PERCEPTUAL PHENOMENON

From the map reader's point of view, the reason for line generalization is to allow a geographic feature to be recognized in cartographic representation, to use a map for the purpose for which it was intended. The importance of feature identification on thematic maps has long been a major focus in the study of shape generalization (Dent 1972: Muehrcke 1969; Dobson 1980; Prince 1977; Turner 1977; Duncan 1981) and in studies of map complexity (MacEachren 1982; Monmonier 1974; Lavin 1979; Muller 1976; Olson 1975). The general line of thought has been that the success of the graphic communication is based on the facility with which the generalized feature is recognized as a specific geographic feature. It would appear, then, that line character includes a definite perceptual component.

Accuracy, too, is an important part of the recognition process. Consider, for example, the importance of accurate coastline recognition for a sailor navigating in a triangle race around the islands of Puget Sound; or the measurements which a tactical pilot makes in planning his approach path for a high-speed reconnaissance flight; or the accurate comparisons involved in cruise missile feature recognition. Accuracy can be retained during generalization by prohibiting coordinate movement for some or all of the original points. Preservation of perceived recognizability may be less directly achievable. For the map reader, generalization should be considered as a transformation which preserves a delicate balance between measurable accuracy and recognizable character.

As a form of picture abstraction, the process of map generalization is probably a simple case of a more general problem of pattern analysis, which in turn may be considered to be the basis for the important inductive approach to any knowlege. *(Tobler 1966, p. 1)*

The basis of pattern analysis is pattern recognition, or the identification of objects and features. It is based to a degree on information theory (Shannon and Weaver 1949), which defines information probabilistically as the reduction of alternative outcomes. Availability of more outcomes in a situation implies more available information; for a continuous phenomenon, the maximum amounts of information will be available at those 'events' in which the phenomenon is changed.

The cartographic application is clear: for surfaces, 'events' are measured as slope or shading (color), and the inflection points or most abrupt change in surface shading will be the most informative points on the surface. For lines, which are topologically one-dimensional, informative events are defined by a change in direction of the line. Perceptually speaking, then, visual information in a graphic display is concentrated along contours or sudden breaks in elevation or shading, and along these breaks, information is concentrated at points of angular inflection (Attneave 1954). It seems logical to assume that to preserve perceptual information on a line, one would insure that the points of angular inflection are preserved during the generalization process.

Two cartographers have tested this, in isolated studies: it is interesting how similar are their verifications, given their different research designs. Marino (1979) presented two groups of people with a series of cartographic lines, one at a time, accompanied by a box of dressmaker's pins. One group had previous experience in compilation and generalization tasks, while the other did not. People were asked to put pins in the lines at points which they felt should be retained in a simplified version of the line, in order to preserve its visual character. No time limit was set on the task.

The experiment was repeated three times; in each successive trial, the same line was presented, but with only half as many pins available. Marino found a significant consensus in the set of points chosen for each line, and further, found no significant differences in response between the two groups. She concluded that cartographic training has no apparent effect in determining which points are critical to recognition of line character; and also that there is a hierarchy in the definition of line character, in that points chosen for the trial with fewest pins were often chosen in all three trials. The major shortcoming of her study is that Marino never summarized angular deviations to determine whether her trial lines had distinguishable angular characters, but rather made an a priori assumption that this was the case.

Kelley (1977) was interested in functional as well as scale generalization. Functional generalization does not require a reduction of information expressly for reduction of map scale, but rather involves the emphasis and/or deemphasis of features when generalizing between maps to serve different purposes (as in collecting information for a thematic map from a topographic data base). Kelley follows a fairly strict interpretation of the Shannon-Weaver (1949) information-

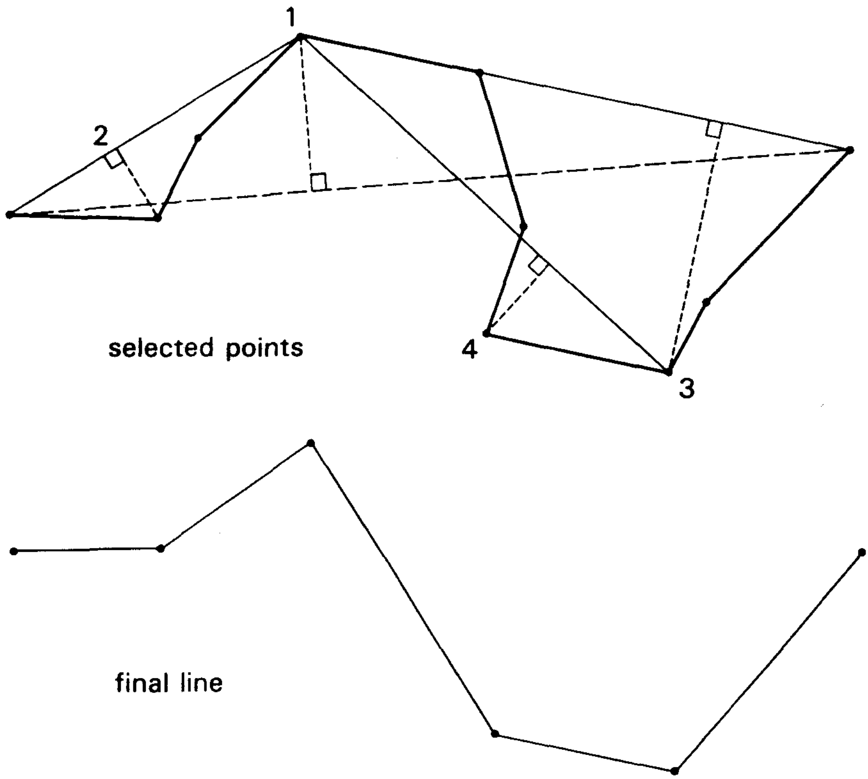

FIGURE *15*. *Line reduction algorithm (after Douglas nd Peucker, 1973).*

theoretic model, assuming that line continuance has an associated probability function which is inverse to the severity (degree) of angular change. He tests the hypothesis that information which is important to the map reader will be found near points of maximum angular inflection.

Two points distinguish his experimental design from Marino's. First, Kelley embeds his trial lines in a map context, and texts for the constraining effects of background noise, familiarity with the mapped feature, and the closure of the lines. Second, he measures the rate of changing direction along all lines tested, and thus determines where points of maximum angular inflection occur. Analysis is fairly direct, and involves comparing locations chosen by perceptual response to locations computed mathematically. Kelley finds a significant correlation between the two, and a strong consensus in point choice as well; neither of these is affected significantly by closure or by background noise. Kelley's findings on familiarity are weak, and he is not confident of his interpretations.

It would appear that these studies confirm the work of Attneave (1954), mentioned earlier, that information critical to the recognition of a line can be found in the vicinity of points of maximum angular deviation, and further, that most people will choose the same set of points along a line as being critical to its recognition. The conclusion is that a generalization algorithm which consistently selects these critical points will produce a more easily recognizable cartographic line.

The Douglas algorithm (Douglas and Peucker 1973) provides a good example of this, as it tends to select points of maximum angular deviation. The algorithm has been tested informally, by using this routine to generalize Marino's lines (Jenks 1982, personal communication). The set of points which results from this generalization bears similarity to the set of points chosen consistently by Marino's test subjects.

The mechanics of the algorithm follow from Peucker's ideas on bandwidth filtering, as described in a previous section of this chapter. Basically, the endpoints of the line to be generalized are connected by a straightline segment (Figure 15), and perpendicular distance from this segment is measured to each point. The point which is farthest from the segment is stored. This becomes a new endpoint for a subsegment (actually, two subsegments – one to each of the previous endpoints), and the process continues until a threshold distance between all points and the straightline segments is reached. Incidentally, each defined segment identifies a specific bandwidth; as discussed earlier, the concepts developed in this algorithm formed the basis for Peucker's bandwidth theory of the cartographic line.

SUMMARY

One can see that there are widely divergent viewpoints on how to measure a cartographic line, and how to construct its valid generalized representation. What this paper has intended to establish is that common to all approaches to generalization is the stumbling block caused by lack of a consistent definition of line character. The problem may be similar in difficulty to the problems of measuring length and sinuosity: it may be that no unique solution exists. But in reviewing this body of research, an approach does seem to present itself, if only in rough form.

Linear information is composed of two types of components. One is tied to the accuracy of the line. Geographic accuracy involves careful positioning of coordinates. The limits of positional accuracy are determined by the scale at which the line is measured; and points selected for positional accuracy are related to the geographic context at hand. For example, city locations should be carefully positioned on the land-side of a coastline or river bank. Other points should also be carefully positioned along the line, in part to anchor the trend of line direction to as large a point set as possible. Equally important, however, the generalized version of the line should retain its visual clues for easy recognition. Perceptual accuracy is related hierarchically to the angular deviations along the line, with maximum importance placed at points of maximum inflection. Thus both kinds of accuracy are components of line information; and can be directly constrained to anchor the trend of a line in cartographic representation.

The other components of linear information are more statistical in nature, and not directly tied to absolute coordinate position. One component is visually recognizable as the characteristic twists and turns which ascribe a generic type of feature to a cartographic line ("oh, yes, that's a fjord, and this, over here, is a meandering stream"). This has been called line character, or visual or graphic structure, depending on the author. Attempts to describe this component have proved largely inadequate, partly because (as with Maling's sinuosity) the defining measures as constructed have been tied to geographic accuracy; the associated difficulties have been discussed at length in this paper.

A more promising approach to defining line character comes in light of two notions, first that line character is statistical, and secondly in development of methods which introduce stochastic information to a line. Partial success of these can be seen in construction of fractal lines and surfaces, which bear strong visual resemblance to geographic features. The major criticism of fractal modelling is that scale-related cues for recognition are retained at only one level of resolution; at all other levels, the same set of visual cues is reiterated. Allometric models may also provide good representations, without the limitations imposed by these assumptions of self-similarity (Gould 1966).

In either of these techniques, some of the visual cues which one expects to find in a graphic representation are simulated; what results is an enhancement of existing information, rather than a conventional generalization. Because certain aspects of linear character can be modelled, cartographic representations of visual characteristics are possible which mimic real world geographies. But the ability to reconstruct a cartographic line or surface given its particular visual cues is somewhat less robust than the ability to generate a line or surface given only a description of what visual characteristics ought to be present to recognize it as a fjord, for example, or as a sandy beach, at a series of scales of representations.

This is not to say that the work on stochastic modelling will not be useful for cartographic application, but rather to reiterate the need for objective identification procedures for describing the aspects of line character. Once these have been developed, and tested, the enhancement of cartographic lines and surfaces may afford greater flexibility in automated generalization with reduced storage requirements: for example, Carpenter's (1981) image of Mount Rainier was constructed using only 100 coordinates. There are at present no articles in the professional literature reporting perceptual evaluation of these kinds of stochastic enhancements, and it is difficult to presuppose the extent to which linear enhancement can be relied upon to produce consistently recognizable generalizations. It is hoped that the present review has provided a framework within which to understand how these statistical representation techniques have developed, and how they may be applied in the context of generalization.

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RESUME Traditionnellement la généralisation cartographique a été considérée comme un allègement des détails. Cependant, dans le but d'aider le repérage visuel, en certains cas récents on a ajouté des détails pour accentuer le message cartographique. La technique d'accentuation est de plus en plus utilisée dans l'affichage des lignes, surfaces et paysages générés par ordinateur; les représentations fractales sont parmi les exemples les mieux connus de cette technique. Pour justifier l'accentuation on invoque surtout l'efficacité accrue avec laquelle de tels affichages peuvent être faits; à l'encontre, on cite les caractéristiques visuelles auxquelles le lecteur s'attend. On reconsidère les tendances en recherche cartographique pour expliquer comment s'est développé le changement entre la généralisation par allègement et celle par accentuation.

ZUSAMMENFASSUNG Im Überlieferten Sinn wurde die kartographische Generalisation mit dem Weglassen von Einzelheiten gleichgesetzt. Neuerdings hat man jedoch Kartenelemente hinzugerfügt, um die Kartenaussage zu erhöhen, und als Hilfsmittel zum visuellen Verständnis der Karte. Derartige Verbesserungen sind zunehmend gebräuchlich bei der Darstellung von computererzeugten Linien, Flächen und Diagrammen: einige der besser bekannten Beispiele dieses Verfahrens sind fraktale Darstellungen. Die Rechtfertigung für diese Methode liegt in der effektiven Art, mit der man solche Darstelungen erzeugen kann; die Gegenargumente kreisen um die visuellen Merkmale, die der Kartenleser erwartet. Der Verfasser präsentiert eine Betrachtung von Trends in der kartographischen Forschung und erläutert, wie sich der Wechsel im Generalisieren vom 'Weglassen' zum 'Hinzufügen' entwickelt hat.

RESUMEN Tradicionalmente, la generalización cartográfica ha sido considerada en el contexto de eliminación de detalles en el mapa. En situaciones recientes, sin embargo, se han introducido detalles para realizar el mensaje del mapa como una ayuda en el reconocimiento visual. Los realces son más y más prevalentes en despliegues de líneas generadas por computadora, superficies y paisajes; algunos de los ejemplos más conocidos de esta técnica son las representaciones fractales. La justificación citada para este realce es la mejora en la eficiencia con que pueden construir exposiciones; los argumentos en contra mencionan las características visuales que el lector del mapa espera ver. Se presenta una reconsideración de las tendencias en la investigación cartográfica para explicar como ha desarrollado el cambio en la generalización de 'eliminación' a 'realce'.