Mathematical Geology, Vol. 12, No. 2, 1980

Fractals and the Accuracy of Geographical Measures[1]

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The problems of estimating line length, area, and point characteristics in the earth sciences have generated substantial but independent literatures. All three problems are of increasing concern given the current interest in digital capture, processing, and the storage of geographically referenced data. In the case of qualitative maps, all three are shown to be related to Mandelbrot's fractional dimension D *(Mandelbrot, 1977) which allows the dependence of each on sampling density to be predicted. The general results are confirmed by simulation on surfaces of constant* D. *They also imply that certain improvements can be made in a number of previously proposed methods.*

KEY WORDS: Fractals, spatial distributions, map analysis.

INTRODUCTION

The problem of defining the length of a coastline, lakeshore, or topographic contour is a recurring theme in a number of spatially oriented disciplines. A recent paper by Håkanson (1978) recognized that the length of such geographic lines cannot be defined in a simple manner without also specifying the measurement method. If measurement is made from a map, length depends on the degree of generalization of the map, which is a function of scale. On the other hand, if length is measured on the ground, a scale factor is involved through the sampling interval necessarily inherent in the method of measurement. Håkanson's solution was to propose a method for determining an appropriate scale of measurement, so that independent investigators studying the same line would at least arrive at the same length estimate, albeit still scale-specific. He also developed an empirical expression for the variation in length over scale which can be compared to a number of earlier and similar attempts in the cartographic literature (for a review see Maling, 1968, and more recently, Beckett, 1977).

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0020-5958/80/0400-0085$03.00/0

A mathematical analysis of locally complex map boundaries has been carried out by Matheron and others (see for example Matheron, 1967, pp. 30-41). If boundaries are assumed to be created by a directionally isotropic and spatially stationary stochastic process, then it is possible to relate certain indices, such as the expected length of boundary per unit area, to the first derivative of the spatial autocorrelation function at zero distance.

The approach used in this paper is based on the work of Mandelbrot, which is summarized in a nontechnical way in Mandelbrot (1977) and which provides the conceptual framework for this paper. The next two sections define two of the central concepts—fractional dimensionality and self-similarity—as they are used in this paper, but are not intended to introduce the reader to the very broad scope of Mandelbrot's contribution.

FRACTALS

Richardson (1961) made an extensive study of the relationship between geographic length and scale in connection with his research into geographical causes of conflict. He measured line lengths by sampling points at fixed intervals, and found a tendency for the sampling interval to be related to the length estimate by a power law, that is, as a straight line on double log paper. Mandelbrot (1977, and in particular, 1967) provided a framework for these results by introducing the concept of fractional dimension, as follows.

Suppose that a particular line's length is measured using two different sampling intervals λ_0 and λ, and is found to contain n_0 and n such intervals, respectively. The two length estimates are $n\lambda$ and $n_0\lambda_0$; to each interval of length λ_0, there correspond n/n_0 intervals of length λ. Mandelbrot defines the dimensionality of the line as $D = \log(n/n_0)/\log(\lambda_0/\lambda)$. If the line behaves as a smooth curve of fixed length, then, when the sampling interval is halved, the number of such intervals will double and D will be 1.0, the conventional dimensionality of a line. But for a geographic line, the number of intervals will tend to more than double and D will be greater than 1. The limit to D is provided by lines that grow with scale so as to eventually fill the space, and thus have a dimensionality of 2.0. Geographically, this would correspond to a lake with a shoreline so contorted that it fills the entire area of the lake. Mandelbrot (1977) coined the term fractal to refer to curves and surfaces with noninteger dimensionality.

Richardson's graphs plot the length versus sampling interval, that is, $n\lambda$ against λ. Rearranging the definition of D above, we have $\log n\lambda = a + (1 - D) \cdot \log \lambda$ where a is constant. Thus the straight lines observed by Richardson indicate that D is a constant parameter of the lines he analyzed. The D values range from near 1 in the case of the relatively smooth coast of South Africa to 1.25 for the west coast of Britain.

SELF-SIMILARITY

Self-similarity is defined as a property of certain curves where each part of the curve is indistinguishable from the whole, or that the form of the curve is invariant with respect to scale. A corollary of this property is that it should not be possible to determine the scale of the curve from its form. Furthermore D should be a constant over all scales. In the case of irregular curves such as coastlines, self-similarity must be interpreted statistically—that is, each part of the curve should be statistically indistinguishable from the whole.

Although Richardson's data clearly support the existence of self-similarity, in realty the application of the concept to the natural landscape is limited. Although the west coast of Britain showed a constant D of 1.25 over sampling intervals between 10 and 1000 km, a similar analysis of the east coast by the author showed a variation from 1.15 to 1.31 (Fig. 1). Similarly, a geomorphologist can usually guess limits to the range of possible scales of a coastline, indicating that self-similarity only applies within those limits. Geomorphic processes tend to operate at specific scales, and provide visual clues to the determination of scale. It is interesting that such clues are often misleading when applied to extraterrestrial landscapes, where the landscape-forming processes may be entirely different. The result is that Mandelbrot's simulated self-similar landscapes (see, for example, Mandelbrot, 1977, p. 210) are often more suggestive of lunar terrain.

The notion of a constant D, or self-similarity, in the natural landscape occurs repeatedly in Mandelbrot's work (for example, see Mandelbrot's abstract, 1975a) and has been rejected by a number of geomorphologists. Håkanson (1978) followed Scheidegger (1970) who argued that empirical self-similarity should be rejected as both premature in view of Richardson's limited evidence, and as unacceptable in principle. But self-similarity is only one aspect of the

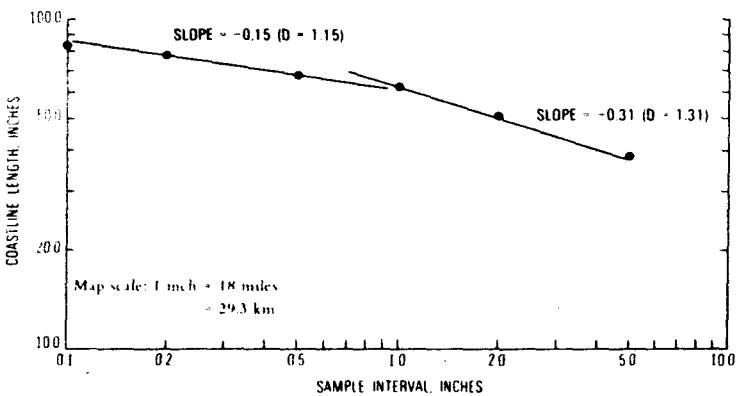

Fig. 1. Fractional dimensionality of the east coast of Britain.

fractal approach, and it would be unwise to reject the entire concept. The position adopted in these pages is that the performance of D as a variable over various scales is a useful index of a real line. It shows the variation in estimates of length and a number of other properties over scales, and the ranges of scale over which the curve is self-similar.

HÅKANSON'S ANALYSIS

Håkanson defined the length of a line by overlaying a grid and counting intersections. Since the grid lines are a constant 0.5 cm apart, the scale of the map can be used as a type of sampling interval. Figure 2 shows his estimated

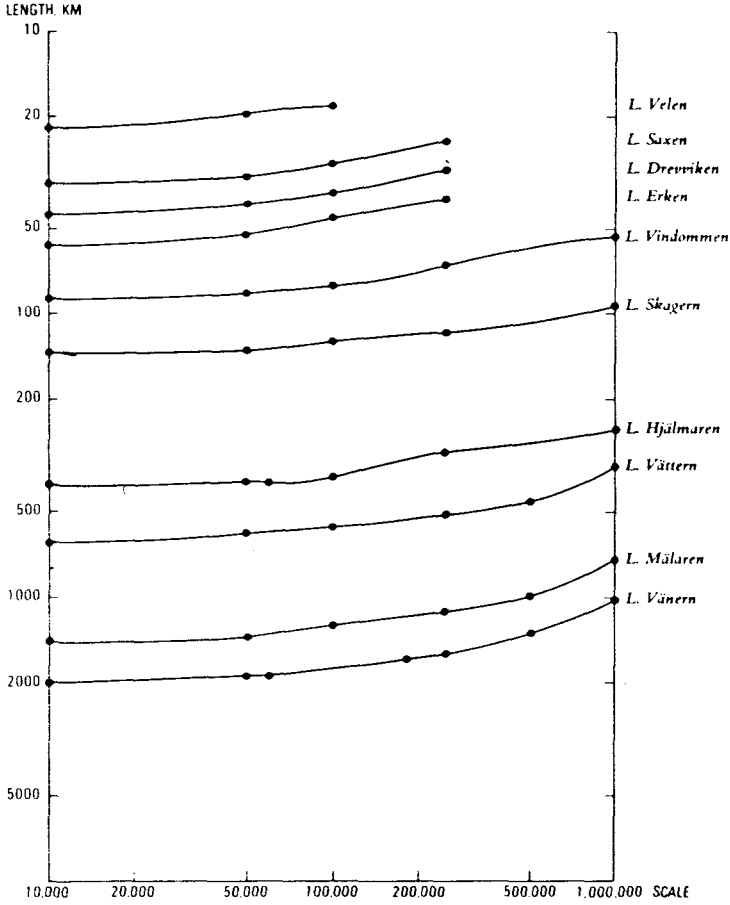

Fig. 2. Håkanson's data for the shoreline lengths of Swedish lakes against scale.

lengths plotted against scale; for a self-similar geographic line the slope of the graph is expected to be $D - 1$. It is clear that the slope of each line increases with scale; the most rapid change of line length with sampling interval occurs on the most generalized maps in each case. At the finest resolution, on the other hand, the dimensionality comes close to 1.0, indicating that at this scale the shorelines approximate smooth curves.

These results are open to various interpretations. The length estimates were made from maps constructed independently at each scale. If we assume that the degree of generalization incorporated in the drawing of lake shorelines on each map was roughly similar, then the variation in D between scales can be interpreted as due to changes in the statistical nature of the shoreline. For example, at coarse resolution (1:200,000) shorelines are more contorted than at fine resolution (1:10,000). On the other hand, it is possible that cartographic generalization varied from map to map; under this interpretation the low D values at fine resolution are due to a greater degree of cartographic generalization.

Finally, Håkanson's estimates are not strictly suitable for an analysis of D. The method relies on counting the intersections between a grid and the shoreline of interest, and while in general we would expect two intersections for each intersected grid cell, in principle any even number of intersections is possible. Thus the scale of the map, which determines the size of the grid, is not strictly related to a sampling interval. It would be more appropriate, in terms of an analysis of fractional dimensionality, to count the number of cells intersected, rather than the number of intersections. For example, suppose two maps are available at a scale of 1:100,000. One has been drawn at that scale, the other by photographically reducing a 1:10,000 map. Two pairs of length estimates are made from a 0.5-cm grid, one pair by counting intersections, the other by counting cells intersected. Then the latter pairs will be more similar than the former. In other words, counting cells intersected gives estimates that are more simply related to scale, and less to how the map was constructed. The next sections introduce two other estimation problems.

AREA ESTIMATION

Two other forms of estimation from a grid overlay are related to Håkanson's method of length measurement. A common method of estimating the area of a closed figure on a map is to overlay a grid, and count the number of cells occupied by the figure. "Occupied by" can be interpreted in two ways—a cell can be counted if its centroid falls within the figure (the centroid rule) or if more than 50% of its area lies in the figure (the majority rule). Clearly the estimate will be greatly improved if the area of each cell intersecting the figure is measured, but this is far more time consuming.

This method of area estimation is commonly used in a number of fields as an alternative to planimetry. The grid may take the form of an array of square

cells, or of regularly spaced points. Several papers have approached the problem of determining the accuracy of the estimate, which depends on the proportion of counted cells that lie on the boundary of the figure. Let N represent the number of cells counted, n the number on the boundary, and b the length of each cell side. The area estimate is thus Nb^2, and the problem is to determine the standard error σ of this estimate.

If a cell which is intersected by the boundary is counted by either rule as lying within the figure, it contributes to an overestimation of the area by an amount equal to the portion of the cell lying outside the figure. Similarly a boundary cell which is not counted contributes to an underestimation. Frolov and Maling (1969) estimated this portion by assuming that the boundary line could be regarded as a straight line drawn randomly and independently across each cell. They found the mean square area of the cutoff portion, that is, the error variance in the area estimate for each boundary cell, to be αb^4, $\alpha = 0.0452$. Lloyd (1976) pointed to a problem with the stochastic process used by Frolov and Maling to define a random line, and recomputed α as 0.0609. In an independent paper Goodchild and Moy (1977) obtained 0.0619 by integrating the same expression (compare Appendix 1 of Goodchild and Moy, p. 79, with Lloyd's expression, p. 25).

The error variance in the estimate of area depends on the summation of individual errors from each of the boundary cells. In the case of a very contorted line, the contribution of each cell may be assumed to be independent, so that the total error variance will be simply the sum of the individual values, or $n\alpha b^4$. The value of α was computed above; on the assumption that each cell is intersected in a straight line, which is incompatible with independence, but it is assumed that the true value of α will be approximately the same for a contorted line.

If the line is in fact smooth, indicating that the straight line model is approximately true, we will find a strong serial autocorrelation between the contributions of neighboring cells. The effect will be to increase the error variance as follows

$$\sigma^2 = ns^2 + 2s^2 \sum_i \sum_{j>i} r_{ij}$$

where σ^2 is the error variance in the area estimate, s^2 the error variance per cell, and r_{ij} the correlation in the contributions of cells i and j. Correlation will likely be strongest when i and j are adjacent. The upper limit of r_{ij} is 1, and the maximum value of σ^2 is therefore $ns^2 + 2n(n-1)s^2/2 = n^2s^2$, which occurs when the contributions are perfectly correlated.

More generally, write

$$\sigma^2 = n^\beta s^2 \qquad \text{where} \qquad \beta = \log \left[n + 2 \sum_i \sum_{j>i} r_{ij} \right] \Big/ \log n$$

Serial independence corresponds to the case $\beta = 1$, and perfect correlation to $\beta = 2$. Thus β can be interpreted as a function of the smoothness of the line. The standard error of an area estimate is therefore $\sigma = \alpha^{1/2} n^{\beta/2} b^2$.

Frolov and Maling (1969) estimated n by relating it to the area of the outline S, through the equation $n = K_3 S^{1/2}/\gamma b$, where γ is a constant equal to the root mean square length of a random straight line drawn across a unit square. Lloyd's correction gives a value of 0.862 (Goodchild and Moy computed the mean length: 0.794). K_3 depends on the shape of the figure; $K_3/2(\pi^{1/2})$ gives the number of cells intersected by the figure relative to the number that would have been intersected by a circle of the same area.

Substituting for n in the expression for σ above, we obtain

$$\epsilon = \alpha^{1/2}(K_3/\gamma)^{\beta/2} S^{(\beta/4)-1} b^{2-(\beta/2)}$$

for the standard error as a proportion of the area estimate. In terms of the number of cells counted, $N = S/b^2$, the error is $\epsilon = \alpha^{1/2}(K_3/\gamma)^{\beta/2} N^{(\beta/4)-1}$. For a constant cell size, ϵ depends on area to the power $(\beta/4) - 1$; for a constant area, it depends on cell size to the power $2 - (\beta/2)$. Frolov and Maling followed the independence assumption which implies $\beta = 1$, and provided empirical evidence to support $\alpha - \frac{3}{4}$ power dependence of error on area as did Moy (1977) and Goodchild and Moy (1977). Tobler (1974) carried out gridding experiments on a number of maps, and published estimates of standard error as a function of cell size. A double log regression analysis of this data gave slopes of -0.733 and -0.801 for Tobler's Tables I and II, respectively, again in good agreement with the theoretical expectation of $-\frac{3}{4}$.

POINT ESTIMATION

The problem to be introduced in this section is that of determining whether a point lies inside or outside a boundary from a grid cell representation of the boundary. Part of the reason for renewed interest in the problems of estimating such map properties as length and area from grids lies in their practical importance in geographic data processing and information systems. Grid representations appear in remote sensing as pixels and as sample heights in digital terrain models, to cite two of the more important application areas. The advantages of a cellular representation lie in the relative ease of digitizing, processing, and display operations over alternatives such as the polygon representation (for a current commentary on this controversy see Tomlinson, Calkins and Marble, 1976). In such applications it is important to know the loss of information and accuracy inherent in a grid, and its relationship to cell size.

The point estimation problem is to determine the probability that the characteristics at a randomly chosen point, as determined from a grid representation, are the correct characteristics. This is the probability that the cell centroid (for

the centroid rule) or the majority of the cell (for the majority rule) lies within the same map zone as the point. For a single zone, the probability π_1 that a point in the zone will be misclassified as belonging to some other zone is given by the total of the overestimating portions of boundary cells as a proportion of zone area; conversely, the probability π_2 that a point in some other zone will be misclassified as belonging to this zone depends on the underestimation portions.

The relationship between these probabilities and the previous analysis is straightforward. For those boundary cells which contribute to an overestimation of area, let δ denote the mean fraction of cell area contributed. One half of the boundary cells are expected to contribute to overestimation, giving a total area of $n\delta b^2/2$. Thus $\pi_1 = n\delta b^2/2S$. Using the same argument for those cells contributing to underestimation and substituting for n gives

$$\pi_1 = \pi_2 = (K_3\delta/2\gamma)\, bS^{-1/2} \qquad \text{or} \qquad (K_3\delta/2\gamma)\, N^{-1/2}$$

Misclassification probabilities thus decrease with the number of cells used. Moy (1977) found that empirical evidence supported this analysis.

In the next section we examine the preceding discussion in the light of fractional dimensionality.

FRACTALS IN AREA AND POINT ESTIMATION

Both area and point problems depend on an estimate of n, the number of cells intersected by the figure boundary line. This parameter is directly related to the Håkanson length estimation method, and is identical if the number of cells intersected is used instead of the number of intersections. The cell side b is a sampling interval, so n and b can be related through a fractal D:

$$D = \log (n/n_0)/\log (b_0/b)$$

Rearranging, we have $n = ab^{-D}$. The Frolov and Maling analysis implies $D = 1$, in other words, that each boundary line behaves as a smooth curve. The difference lies in the K_3 or shape factor; by assuming it to have a constant value, we assume in effect that shape is the same over all scales, whereas the shape of a curve of $D > 1$ will grow more complex as cell size is reduced, or resolution increased.

Substituting for n, we now have $\epsilon = \alpha^{1/2} a^{\beta/2} b^{2-(D\beta/2)} S^{-1}$. For a fractal curve of dimension 1.5, for example, ϵ will depend on cell size to the power 1.25, assuming $\beta = 1$. The more contorted the boundary, or the higher its dimensionality, the less rapid the increase in error with cell size.

Taking a similar approach to the point estimation problem, we find that the mismatch area for a figure $\Delta = n\delta b^2 = \delta a b^{2-D}$. The misclassification probabilities are $\pi_1 = \pi_2 = (\delta a/2)\, b^{2-D} S^{-1}$.

SIMULATIONS

In this section the relationships developed above are verified by simulation. A boundary of prescribed dimensionality D can be generated by taking an arbitrary contour of a surface of dimensionality $D + 1$. Let $z(x, y)$ and $z(x + u, y + v)$ denote the elevations of a surface at (x, y) and $(x + u, y + v)$, respectively, and write $d = (u^2 + v^2)^{1/2}$. Then consider the expected squared difference between the elevations at pairs of points the same distance apart. The expectation as a function of distance is termed the variogram of the surface.

$$\gamma(d) = E[z(x, y) - z(x + u, y + v)]^2$$

Consider the class of variograms $\gamma(d) = d^{2H}$. Orey (1970) showed that such "fractional Brownian" surfaces have profiles, and by extension contours, of dimension $2 - H$.

Mandelbrot (1975b) showed that it is possible to simulate surfaces of arbitrary H, and thus arbitrary D (Mandelbrot, 1975a). The surfaces generated by the author in Figure 3 show dimensionalities from 2.1 to 2.7 in steps of 0.1, corresponding to contour line dimensionalities from 1.1 to 1.7.

Each of the surfaces shown in Figure 3 was generated using a 100 X 100 sample grid. Since each one has the self-similarity property, or constant D, tests were conducted to demonstrate the relationship between cell size and estimation. A contour was drawn on each surface at a height equal to the mean of the minimum and maximum heights. Cells were aggregated into larger aggregates of 5 X 5, 7 X 7, 9 X 9, and so on, up to 19 X 19, and classified as black or white depending on whether the central cell lay above or below the contour, respectively (the centroid rule).

To estimate the number of boundary cells n as a function of aggregate size b, a count was made of the number of black aggregates containing at least one white cell, in other words, cut by the boundary. Table 1 shows the result of

Table 1. Simulations Based on Self-Similar Surfaces of Various D

		Regression slopes			
		Boundary cells: n		Point estimation: π_2	
H	D for contour	Expected	Observed	Expected	Observed
0.3	1.7	-1.7	-2.01	0.3	0.25
0.4	1.6	-1.6	-1.68	0.4	0.25
0.5	1.5	-1.5	-1.32	0.5	0.51
0.6	1.4	-1.4	-1.30	0.6	0.89
0.7	1.3	-1.3	-1.07	0.7	0.79
0.8	1.2	-1.2	-0.96	0.8	0.83
0.9	1.1	-1.1	-1.27	0.9	1.10

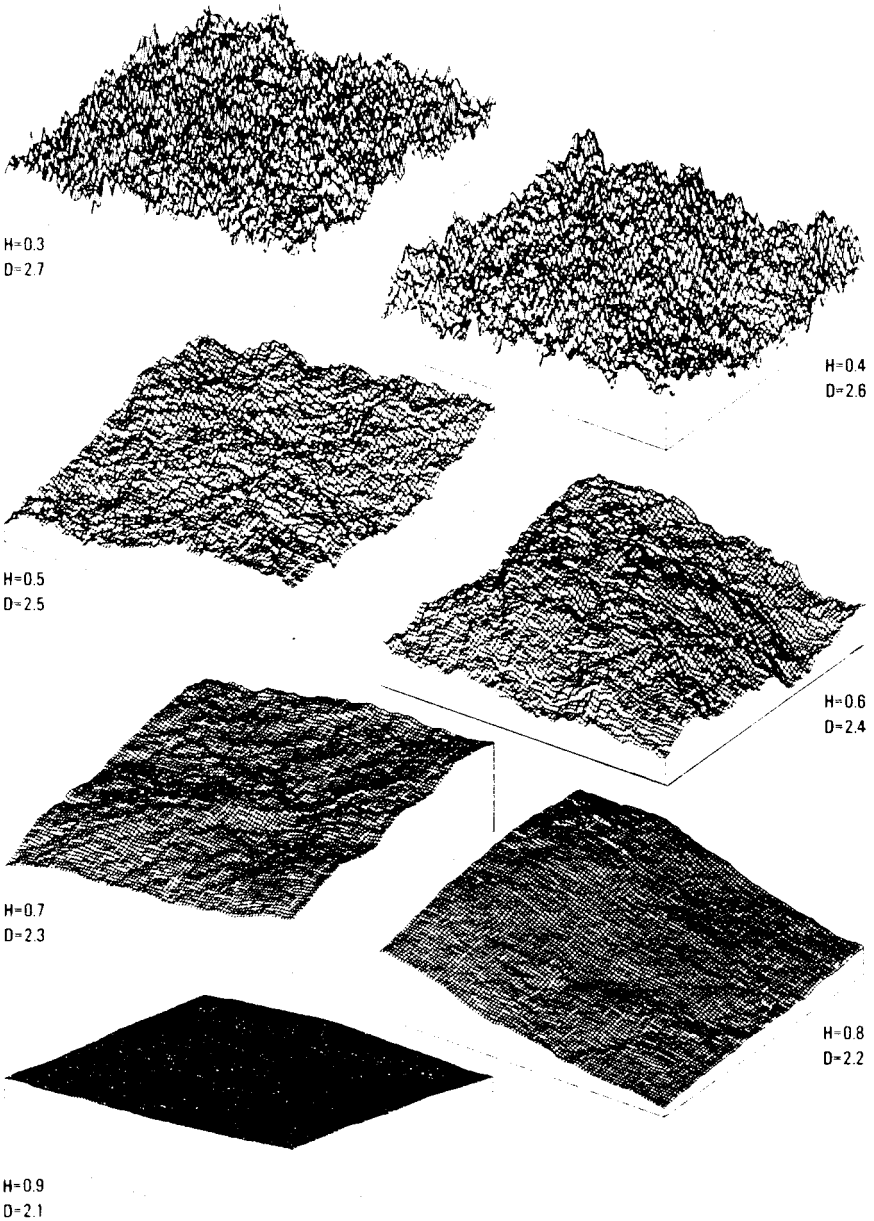

Fig. 3. Self-similar surfaces generated by the fractional Brownian process with variable H.

regressing log n against log b, and the slopes show general agreement with the expected value of $-D$ based on the simulated surface's dimension, despite an obviously substantial standard error of estimate.

The dependence of point estimation on D was examined by estimating π_2, the number of white cells within black aggregates as a proportion of the number of black cells. For this analysis the range of b was from 5 to 33 in steps of 2; the small number of aggregates for high b is less of a problem in this case than in the previous one. The table shows the slopes obtained by regressing log π_2 against log b, and again there is broad agreement with the predicted value of $2 - D$. The area estimation problem was not simulated because of the difficulty of performing an analysis over several repeated griddings of the same surface.

MISMATCH ESTIMATION

Switzer (1975) presented an entirely different approach to the general problem of grid estimation. Mismatch area is defined as the percentage of total area lying within one boundary but in a grid cell counted as lying outside the boundary. It is clearly equal to the sum of the underestimating portions of the boundary cells, in other words $\pi_2 S$. In this section Switzer's analysis will be reviewed in order to place it within the same framework. The notation has been modified somewhat to make it compatible with the rest of the paper.

π_2 was defined above as the probability that a randomly chosen point within the grid representation of a figure does not in fact lie in the figure—briefly, that a point in a black cell is in fact white. It is an estimate of the total white area within black cells.

Write $P_2(d)$ as the probability that a randomly chosen point in a black cell, distance d from the center of its cell, is white. Then we have

$$\pi_2 = S^{-1} \sum_{h=1}^{N} \int_{A_h} P_2(|s - s_h|)\, d\mu(s)$$

where the summation is over each of the N black cells. The integral is over the area of cell h, and s_h denotes the location of the central point. $|s - s_h|$ is therefore the distance between the center and every other location in the cell.

Switzer proposes that $P_2(d)$ can be approximated by a Taylor expansion in its derivatives with respect to d. Taking the first two terms, and recognizing that $P_2(0) = 0$, we have

$$P_2(d) = P_2' d + \tfrac{1}{2} P_2'' d^2$$

where P_2' and P_2'' are the first two derivatives evaluated at $d = 0$. Thus

$$\pi_2 = (P_2' p_1 + \tfrac{1}{2} P_2'' p_2)$$

where p_1 and p_2 are the mean and mean square distance of a random point from the cell center, respectively.

Swtizer shows that P_2' and P_2'' can be estimated from the pattern of white and black cells in the following way. Let $m(1)$ denote the number of times a black cell can be found with a white neighbor (defining neighbor as the Rook's case again). If we ignore cells on the edge of the map each black cell has four chances of having a white neighbor and may be counted up to four times, therefore. Similarly let $m(2)$ denote the number of times a black cell has a white neighbor two cells away, again with a potential count of four for each black cell. Using calculated values for p_1 and p_2 for a square cell, Switzer shows that a good estimate of π_2 is given by

$$\pi_2 = [0.60\, m(1) - 0.11\, m(2)]/4N$$

unless the estimate is negative, in which case it should be set equal to zero.

The main source of uncertainty in applying Switzer's analysis is in estimating the derivatives of $P_2(d)$. It has been supported by empirical work by both Switzer and Muller (1978).

The relationship between $m(1)$, $m(2)$, and scale can be predicted from the fractional dimension D of the boundary of the figure as follows. Consider one row of cells. For a given area, the number of black cells in the row is proportional to b^{-1}, and the number with white neighbors to b^{1-D} (Mandelbrot, 1975b). Thus if the contour has dimensionality 1, the number with white neighbors is a constant independent of the number of black cells, but for $D > 1$ it rises with increasing resolution. Since the number of rows containing black cells is proportional to b^{-1}, the total count of black cells with white neighbors, $m(1)$, is proportional to b^{-D}. Similarly $m(2)$ depends on $(2b)^{-D}$. In general Switzer's estimate can be rewritten as

$$\pi_2 = k(D)\, S^{-1} b^{2-D}$$

$k(D)$ is given by $k'[0.60 - 0.11(0.25)^{D/2}]/4$ where k' is the constant of proportionality in the equation $m(1) = k'b^{2-D}$. Switzer's result is thus compatible with the previous analysis for π_2, both in the dependence of π_2 on cell size b, and on the dimensionality of the figure boundary.

For smooth boundaries, with D close to 1, the relationship between mismatch area and cell size is approximately linear. Muller (1978) observed this empirically, but seems to have misinterpreted the relationship between mismatch area and the standard error of estimate ϵ, as he mistakenly concluded that his results disagreed with the predictions of the Frolov and Maling model.

DISCUSSION

The Switzer and Frolov and Maling approaches are very different in application. In Switzer's the accuracy of a point estimate is predicted from an analy-

sis of the grid representation itself, without any reference to any "true" map. On the other hand Frolov and Maling attempt to predict the outcome of a gridding at a particular scale, in terms of area estimation, from the characteristics of the true map. Their analysis therefore includes an examination of the dependence of estimation on grid size, whereas Switzer implies that $m(1)$ and $m(2)$ should be determined independently at each scale. The two approaches also lead to different kinds of predictions: Switzer of π_2 and Frolov and Maling of ϵ.

This paper has demonstrated that both π_2 and ϵ are determined by the number of cells intersected by the boundary, n, which is also the important parameter in length estimation. Thus all three problems of length, area, and point estimation can be placed within the same framework. The performance of n over changes in cell size can be related to a fractional dimensionality parameter of the figure, or if it is a contour, of the surface by extension. The relationships were demonstrated by simulation on self-similar surfaces.

Although D is not expected to be constant in reality except for limited areas and over limited ranges of scale, it clearly provides a very useful summary from which a number of properties can be predicted. Knowing D, we can estimate the relative benefit of using smaller grid cells or denser sampling, or the relative cost of increasing cell size. Given the usefulness of the concept, there is a pressing need for empirical studies to evaluate the precise pattern of variability in the parameter for different kinds of geographical data, and the ways in which it is influenced by geomorphic processes.

ACKNOWLEDGMENTS

I am grateful to an anonymous referee for providing corrected values of the coefficients in Switzer's published expression for π_2. This research was supported by the Natural Sciences and Engineering Research Council of Canada.

REFERENCES

Beckett, P., 1977, Cartographic generalization: Cartographic J., v. 14, p. 49–50.

Frolov, Y. S. and Maling, D. H., 1969, The accuracy of area measurements by point counting techniques: Cartographic J., v. 6, p. 21–35.

Goodchild, M. F. and Moy, W. S., 1977, Estimation from grid data: the map as a stochastic process, *in* Proceedings of the Commission on Geographical Data Sensing and Processing, Moscow, 1976, (R. F. Tomlinson, ed.), Ottawa: International Geographical Union, Commission on Geographical Data Sensing and Processing.

Håkanson, L., 1978, The length of closed geomorphic lines: Mathematical Geol. v. 10, p. 141–167.

Lloyd, P. R., 1976, Quantization error in area measurement: Cartographic J. v. 13, p. 22–26.

Mandelbrot, B. B., 1967, How long is the coast of Britain? Statistical self-similarity and fractional dimension: Science v. 156, p. 636–638.

Mandelbrot, B. B., 1975a, Stochastic models of the Earth's relief, the shape and the fractal dimension of the coastlines, and the number-area rule for islands: Proceedings of the National Academy of Sciences, v. 72, p. 3825–3828.

Mandelbrot, B. B., 1975b, On the geometry of homogeneous turbulence, with stress on the fractal dimension of the iso-surfaces of scalars: J. Fluid Mechanics v. 72, p. 401–416.

Mandelbrot, B. B., 1977, Fractals: Form, Chance and Dimension: Freeman, San Francisco, 365 p.

Maling, D. M., 1968, How long is a piece of string?: Cartographic J. v. 5, p. 147–156.

Matheron, G., 1967, Éléments pour une Théorie des Mileux Poreux: Masson, Paris, 166 p.

Moy, W. S., 1977, Estimation from grid data: the map as a stochastic process. Unpublished M.A. Dissertation, Department of Geography, University of Western Ontario.

Muller, J. C., 1978, Map gridding and cartographic errors: a recurrent argument: Canadian Cartographer v. 14, p. 152–167.

Orey, S., 1970, Gaussian sample functions and the Hausdorff dimension of level crossings. Zeitschrift fur Wahrscheinlichkeitstheorie und verwandte Gebiete v. 15, p. 249–256.

Richardson, L. F., 1961, The problem of contiguity: General Systems Yearbook v. 6, p. 139–187.

Scheidegger, A. E., 1970, Theoretical geomorphology (2nd ed.): Springer Verlag, New York, 333 p.

Switzer, P., 1975, Estimation of the accuracy of qualitative maps, *in* Display and analysis of spatial data, (J. C. Davis and M. J. McCullagh, eds.): Wiley, London, 378 p.

Tobler, W. R., 1974, The accuracy of categorical maps: Department of Geography, University of Michigan, Cartographic Laboratory Report Number 4.

Tomlinson, R. F., Calkins, H. W. and Marble, D. F., 1976, *Computer Handling of Geographical Data*: UNESCO Press, Paris, 214 p.